Programmation 1

TD n°9

17 novembre 2020

Exercise 1 : Booleans

We consider $\mathbf{Bool}_\perp = \{0, 1, \perp\}$ with the usual ordering ($x < y$ if and only if $x = \perp$ and $y \neq \perp$).

1. What are the Scott-open sets of $\mathbf{Bool}_\perp$? What are the Scott-closed ones ?
2. Show all the monotonic functions from $\mathbf{Bool}_\perp$ to $\mathbf{Bool}_\perp$.
3. Let D be a DCPO, and f a monotonic function from $\mathbf{Bool}_\perp$ to D. Show that f is Scott-continuous.
4. Design $\mathbf{Bool}_\perp \times \mathbf{Bool}_\perp$ (product ordering).
5. List the Scott-continuous functions f such that f restricted to $\{0, 1\}$ defines the Boolean function «OR».

Exercise 2 : Topology and separation

1. Show that if the Scott topology over $(X, \leq)$ is *separated*, i.e. for any $x \neq x'$ there are two open neighbourhoods U and U' respectively of x and x' whose intersection is empty ; then $\leq$ is actually equality over X.
2. Show that Scott topology is T_0, i.e. for any $x \neq x'$ there exists an open neighbourhood of x which does not contain x' *or the converse.*

Exercise 3 : Reals with arbitrary precision

Let $I = \mathbb{R}$ and $J = \{[x, y] \mid x, y \in I, x \leq y\}$ with the $\supseteq$ ordering.

1. Show that J is a DCPO. Is it a lattice ? Complete ?
2. Give a monotonic function from J to $\mathbf{Bool}_\perp$ which is not Scott-continuous.
3. What are the maximal elements of J ? We denote M to be the set of maximal elements.
4. Let f be a continuous function from J to $\mathbf{Bool}_\perp$. Show that $f^{-1}(\{1\})$ is open.
5. Consider the mapping $I : x \mapsto \{x\}$ which goes from $[0, 1]$ to J. Show that I is continuous.

Bonus What is the Scott topology restricted to the set of maximal elements ?

6. Show that M is connected, i.e. there does not exist disjoint, non-empty open sets U, V such that
$$U \cap M \uplus V \cap M = M$$
7. Let g be a continuous function from J to $\mathbf{Bool}_\perp$ such that for all $x \in I$, $g(\{x\}) \neq \perp$. Show that g is constant over I.
8. Imagine a programming language that implements reals with arbitrary precision using intervals. How will the equality function of this language be computed ?

Exercise 4 : Optional homework [1]

We consider the language $\{\star, \bullet\}$ equipped with the small-step semantics as shown :

$$X \star \bullet Y \to XY \qquad \text{if } \exists n \geq 0, X = \star^n$$
$$X \bullet \star Y \to XY \qquad \text{if } \exists n \geq 0, X = \bullet^n$$

1. State and prove the determinism theorem.
2. We consider the DCPO $\{0, 1\}$ equipped with the flat ordering and the semantics below :

$$[\![\varepsilon]\!]_2 = 0$$
$$[\![aX]\!]_2 = 1 - [\![X]\!]_2 \qquad \text{if } a \in \{\star, \bullet\}$$

   Show that this semantics is correct with respect to the small-step semantics.
3. Same question for the DCPO of integers equipped with the flat ordering and the following semantics

$$[\![\varepsilon]\!]_{\mathbb{Z}} = 0$$
$$[\![\star X]\!]_{\mathbb{Z}} = 1 + [\![X]\!]_{\mathbb{Z}}$$
$$[\![\bullet X]\!]_{\mathbb{Z}} = -1 \ + [\![X]\!]_{\mathbb{Z}}$$

4. We give the notion of observational equivalence next :

$$A \equiv B \triangleq \forall C[\cdot], C[A] \to^* \varepsilon \iff C[B] \to^* \varepsilon$$

   (a) Show that it is an equivalence relation.
   (b) Are the denotational semantics completely abstract ?

1. Answers will be shared on 1st December