Programmation 1

TD n°9

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Exercise 1 : Booleans

We consider $\mathbf{Bool}_\perp = \{0, 1, \perp\}$ with the usual ordering ($x < y$ if and only if $x = \perp$ and $y \neq \perp$).

1. What are the Scott-open sets of $\mathbf{Bool}_\perp$? What are the Scott-closed ones ?
2. Show all the monotonic functions from $\mathbf{Bool}_\perp$ to $\mathbf{Bool}_\perp$.
3. Let D be a DCPO, and f a monotonic function from $\mathbf{Bool}_\perp$ to D. Show that f is Scott-continuous.
4. Design $\mathbf{Bool}_\perp \times \mathbf{Bool}_\perp$ (product ordering).
5. List the Scott-continuous functions f such that f restricted to $\{0, 1\}$ defines the Boolean function «OR». Furthermore, seeing $\perp$ as "a divergent calculation", give a calculative interpretation of each of the extensions to $\mathbf{Bool}_\perp$ of the Boolean function «OR».

Sketch:

1. The Scott-open sets are $\{0\}, \{1\}, \emptyset, \{0, 1\}, \{0, 1, \perp\}$. The Scott-closed sets are the complements.
2. The monotonous functions from $\mathbf{Bool}_\perp$ to itself are
   (a) All the usual Boolean operations (id, negation, etc) lifted.
   (b) The operations which send $\perp$ to $\perp$, and which send anything to 0 and 1.
   (c) The functions which send all values to $\{0\}$ or to $\{1\}$.
3. Since f is a monotonic function and $\mathbf{Bool}_\perp$ is flat, we can calculate the sup in a straightforward way.
4. Draw the sets and obtain the product.
5. We can either consider the computations to be $\vee$ *eager*, lazy (in both directions) or parallel. And the interesting case arrives when we also include $\perp$ in the domain.

Exercise 2 : Topology and separation

1. Show that if the Scott topology over $(X, \leq)$ is *separated*, i.e. for any $x \neq x'$ there are two open neighbourhoods U and U' respectively of x and x' whose intersection is empty ; then $\leq$ is actually equality over X.
2. Show that Scott topology is T_0, i.e. for any $x \neq x'$ there exists an open neighbourhood of x which does not contain x' *or the converse.*

Sketch:

1. If $x \leq x'$ then in particular they are not separable, therefore, we necessarily have that $x \leq x' \implies x = x'$.

2. Either x and y are incomparable (direct), or they are comparable and wlog $y \leq x$. Then, we know that $x \not\leq y$. Hence, x belongs to the set U_y which is $U_y = \{x \in X : x \not\leq y\}$, but $y \notin U_y$ by reflexivity. It is clear that U_y is an open set, hence, the topology is T_0.

Exercise 3 : Reals with arbitrary precision

Let $I = \mathbb{R}$ and $J = \{[x, y] \mid x, y \in I, x \leq y\}$ with the $\supseteq$ ordering.

1. Show that J is a DCPO. Is it a lattice ? Complete ?
2. Give a monotonic function from J to $\mathbf{Bool}_\bot$ which is not Scott-continuous.
3. What are the maximal elements of J ? We denote M to be the set of maximal elements.
4. Let f be a continuous function from J to $\mathbf{Bool}_\bot$. Show that $f^{-1}(\{1\})$ is open.
5. Consider the mapping $I : x \mapsto \{x\}$ which goes from $[0, 1]$ to J. Show that I is continuous.

Bonus What is the Scott topology restricted to the set of maximal elements ?

6. Show that M is connected, i.e. there does not exist disjoint, non-empty open sets U, V such that
$$U \cap M \uplus V \cap M = M$$
7. Let g be a continuous function from J to $\mathbf{Bool}_\bot$ such that for all $x \in I$, $g(\{x\}) \neq \bot$. Show that g is constant over I.
8. Imagine a programming language that implements reals with arbitrary precision using intervals. How will the equality function of this language be computed ?

Sketch:

1. No J is not a lattice, since it does not have an element $\top$. On the other hand, it is indeed a DCPO with
$$\sup F = \bigcap F$$
2. It suffices to construct a function which produces $\bot$ on all values except $\{0\}$. It is monotonic, but not Scott-continuous.
3. The maximal elements of J are the singletons.
4. $\{1\}$ is an open set, hence, trivial.
5. Consider $I^{-1}(U)$ and observe that it is inaccessible from the bottom. This implies the existence of an open neighbourhood around each point.

Bonus Scott topology restricted to maximum elements is the usual topology of $\mathbb{R}$. For that, it suffices to show that I is in fact a homeomorphism to M.

6. Since M is the image of $[0, 1]$ by a continuous map, M is connected.
7. The image of a connected set by a continuous function is *connected*. However, the only connected sets of $\mathbf{Bool}_\bot$ not containing $\bot$ are $\{0\}$ and $\{1\}$.
8. A reasonable function of equality must be reflexive and differentiate 0 and 1. If it was Scott-continuous, it would be in its two arguments and therefore the function (==) would be consistent in view of the previous questions.

   The equality function is therefore not Scott-continuous, in particular, it does not correspond to a function expressible in the language that interests us.