Programmation 1

TD n°8

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1 Back to basics

Graphes

Un graphe est une paire $\langle V, E\rangle$ où V est un ensemble *fini* et $E \subseteq V \times V$.
On dit que G est un graphe sur X si $V \subseteq X$.

Expressions de graphes

On donne la grammaire abstraite suivante dont les expressions sont notées Expr_X

$$\begin{aligned} e := {} & \mathsf{Empty} \\ & \mid \mathsf{V}x \qquad x \in X \\ & \mid e \oplus e \\ & \mid e \otimes e \end{aligned}$$

On autorisera dans des calculs intermédiaires de la sémantique à petit pas des expressions $\bar{g}$ où g est un graphe. On notera l'ensemble des expressions intermédiaires Expr^+_X.

Frames d'expressions

On donne la syntaxe suivante pour les frames d'expression

$$\begin{aligned} F := {} & \square \oplus e \\ & \mid g \oplus \square \\ & \mid \square \otimes e \\ & \mid g \otimes \square \end{aligned}$$

Où g est un graphe.

Exercise 1 : Semantics of graphs

1. State then prove the progress theorem on small-step semantics.
2. State then prove the determinism theorem on small-step semantics.
3. State the termination theorem, and prove it.
4. (Bonus *) We transform frames to be of the form : $F := \square \oplus e \mid e \oplus \square \mid \square \otimes e \mid e \otimes \square$.
   (a) Show that the semantics is no longer deterministic.
   (b) State the confluence theorem.
   (c) Prove it.

$$\frac{}{\mathsf{Empty} \to \langle \emptyset, \emptyset \rangle}$$

$$\frac{}{\mathsf{V}x \to \langle \{x\}, \emptyset \rangle}$$

$$\frac{e \to e'}{F[e] \to F[e']}$$

$$\frac{g_1 = \langle V_1, E_1 \rangle \quad g_2 = \langle V_2, E_2 \rangle \quad g_3 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle}{\bar{g}_1 \oplus \bar{g}_2 \to \bar{g}_3}$$

$$\frac{g_1 = \langle V_1, E_1 \rangle \quad g_2 = \langle V_2, E_2 \rangle \quad g_3 = \langle V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2 \rangle}{\bar{g}_1 \otimes \bar{g}_2 \to \bar{g}_3}$$

Figure 1 – Sémantique à petits pas

$$[\![\mathsf{V}x]\!] \triangleq \langle \{x\}, \emptyset \rangle \tag{1}$$

$$[\![\mathsf{Empty}]\!] \triangleq \langle \emptyset, \emptyset \rangle \tag{2}$$

$$[\![e_1 \oplus e_2]\!] \triangleq \langle V_1 \cup V_2, E_1 \cup E_2 \rangle \qquad \text{si } [\![e_1]\!] = \langle V_1, E_1 \rangle \wedge [\![e_2]\!] = \langle V_2, E_2 \rangle \tag{3}$$

$$[\![e_1 \otimes e_2]\!] \triangleq \langle V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2 \rangle \qquad \text{si } [\![e_1]\!] = \langle V_1, E_1 \rangle \wedge [\![e_2]\!] = \langle V_2, E_2 \rangle \tag{4}$$

Pour les expressions étendues, on ajoute la règle suivante

$$[\![\bar{g}]\!] \triangleq g \tag{5}$$

Figure 2 – Sémantique dénotationnelle

(d) Assuming the termination of the new system, deduce the existence of a *unique* normal form for expressions.

After this question, we only study the deterministic semantics defined at the start.

5. State the theorems of correctness and adequacy of the two semantics on the graphs.
6. Prove correction.
7. Prove adequacy.
8. Prove the following equivalence in operational semantics. (Operational semantics has the same normal forms when it is not deterministic.)

$$\forall g, x \otimes (y \oplus z) \to^* \bar{g} \iff (x \otimes y) \oplus (x \otimes z) \to^* \bar{g}$$

9. Define a function $\mathsf{map} : (X \to Y) \times \mathsf{Expr}_X \to \mathsf{Expr}_Y$.
10. Calculate the denotational semantics of map on graphs.
11. What is the set of graphs constructed from the expressions Expr_X ?
12. Assuming the existence of a function $V : \mathsf{Expr}_X \to \mathcal{P}_f(X)$ which to a graph expression associates the set of its vertices, describe a function $N_x : \mathsf{Expr}_X \to \mathcal{P}_f(X)$ which to a graph expression e representing a graph $\langle V, E\rangle$ associates the set $\{y \in X \mid (x, y) \in E\}$. Justify that for all expressions $e \in \mathsf{Expr}_X$ such that $[\![e]\!] = \langle V, E\rangle$ we have the equality $N_x(e) = \{y \mid (x, y) \in E\}$.

2 DCPOs

Rappel sur les familles dirigées

Une famille D non vide d'un ensemble $(X, \leq)$ est dirigée si et seulement si

$$\forall (x, y) \in D, \exists z \in D, z \geq x \wedge z \geq y$$

Rappels sur les DCPOs

Un DCPO est un ensemble partiellement ordonné $(X, \leq)$ tel que toute famille dirigée possède un sup. Un DCPO est *pointé* s'il existe un élément minimal.

Exercise 2 : Cartesian Closed Category

Show that the category of DCPOs is Cartesian closed, by going through the following steps :

1. Show that there exists a DCPO $\mathbf{1}$ such that for any DCPO D there exists a unique function continuous from $\mathbf{1}$ to D.
2. Show that if D_1 and D_2 are two DCPOs then $D_1 \times D_2$ with the product ordering is a DCPO.
3. Show that $D_1 \times D_2$ verifies a universal product property (where all the quantifications are on continuous functions).

$$\forall f : A \to D_1, g : A \to D_2, \exists! h : A \to D_1 \times D_2, \pi_1 \circ h = f \wedge \pi_1 \circ h = g$$

4. Show that $A \implies B$ the set of continuous functions from A to B ordered point to point is a DCPO.
5. Show that if A, B, C are DCPOs, then any continuous function $f : A \times B \to C$ transforms into a function $\Gamma f : A \to (B \implies C)$ which is also continuous.
6. Show that a function $f : A \times B \to C$ is continuous if and only if it is continuous in its two arguments.
7. Show that the evaluation map $\Delta : A \times (A \implies B) \to B$ is continuous.

3 Topology

Topologie

Une topologie τ sur un ensemble X est un ensemble de parties de X qui vérifie

1. τ est stable par intersection finie.
2. τ est stable par union quelconque.
3. τ contient l'ensemble X.
4. τ contient l'ensemble $\emptyset$.

On dira alors d'un élément de τ qu'il est *ouvert*. Le complémentaire d'un ouvert est par définition un ensemble *fermé*.

Fonction continue

Une fonction $f : X \to Y$ est *continue* de (X, τ) vers (Y, θ) si et seulement si

$$\forall U \in \theta, f^{-1}(U) \in \tau$$

Topologie de Scott

Soit $(D, \leq)$ un DCPO. Une partie $U \subseteq D$ est appellée un *ouvert de Scott* si et seulement si elle vérifie

1. U est clos vers le haut :
$$\forall x, \forall y. \quad x \in U \wedge x \leq y \implies y \in U$$
2. U est inaccessible par le bas :
$$\forall E \text{ dirigée} \qquad \sup E \in U \implies E \cap U \neq \emptyset$$

Exercise 3 : Scott topology

1. Show that the Scott toplogy is a topology.
2. Show that a closed set of D is closed at the bottom and closed under suprema of directed subsets.
3. Show that $\downarrow x \triangleq \{y \in D \mid y \leq x\}$ is a closed set of D for Scott topology.
4. Show that the continuous functions for Scott topology are the Scott-continuous functions.

Exercise 4 : Finite words . . . or infinite

Let $S = \{0,1\}^\infty = \{0,1\}^* \cup \{0,1\}^\omega$, with the prefix-ordering.

1. Show that S is a DCPO. Is it a lattice ?
2. What are the maximal elements of S ?
3. Let f be a function from S to $\mathbf{Bool}_\perp$ such that :
$$\forall s \in \{0,1\}^\omega \begin{cases} f(s) = 1 & \text{if } s \text{ contains the factor } 0 \cdot 1 \\ f(s) = 0 & \text{otherwise} \end{cases}$$
Show that f is not Scott-continuous.

4. We consider the function $v : S \to J$ defined by :

$$v(b_1 \cdot b_2 \cdots b_n) = \left[\sum_{i=1}^{n} 2^{-i} b_i, \sum_{i=1}^{n} 2^{-i} b_i + 2^{-n}\right]$$
$$v(b_1 \cdot b_2 \cdots) = \left\{\sum_{i=1}^{\infty} 2^{-i} b_i\right\}$$

   What is v for ? Show that v is Scott-continuous. Is it injective ?

5. Let g be a Scott-continuous function from S to $\mathbf{Bool}_\perp$ which is compatible with v :

$$\forall x, y \in S, v(x) = v(y) \to g(x) = g(y)$$

   Show that if $\forall x \in \{0,1\}^\omega$, $g(x) \neq \perp$, then g is constant over $\{0,1\}^\infty$.