Programmation 1

TD n°8

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1 Back to basics

> **Graphes**
>
> Un graphe est une paire $\langle V, E \rangle$ où V est un ensemble *fini* et $E \subseteq V \times V$.
> On dit que G est un graphe sur X si $V \subseteq X$.

> **Expressions de graphes**
>
> On donne la grammaire abstraite suivante dont les expressions sont notées Expr_X
>
> $$
> \begin{aligned}
> e := \ & \mathsf{Empty} \\
> | \ & \mathsf{V}x \qquad x \in X \\
> | \ & e \oplus e \\
> | \ & e \otimes e
> \end{aligned}
> $$
>
> On autorisera dans des calculs intermédiaires de la sémantique à petit pas des expressions $\bar{g}$ où g est un graphe. On notera l'ensemble des expressions intermédiaires Expr_X^+.

> **Frames d'expressions**
>
> On donne la syntaxe suivante pour les frames d'expression
>
> $$
> \begin{aligned}
> F := \ & \square \oplus e \\
> | \ & g \oplus \square \\
> | \ & \square \otimes e \\
> | \ & g \otimes \square
> \end{aligned}
> $$
>
> Où g est un graphe.

Exercise 1 : Semantics of graphs

1. State then prove the progress theorem on small-step semantics.
2. State then prove the determinism theorem on small-step semantics.
3. State the termination theorem, and prove it.
4. (Bonus *) We transform frames to be of the form : $F := \square \oplus e \mid e \oplus \square \mid \square \otimes e \mid e \otimes \square$.
   (a) Show that the semantics is no longer deterministic.
   (b) State the confluence theorem.
   (c) Prove it.

$$\overline{\mathsf{Empty} \to \langle \emptyset, \emptyset \rangle}$$

$$\overline{\mathsf{V}x \to \langle \{x\}, \emptyset \rangle}$$

$$\frac{e \to e'}{F[e] \to F[e']}$$

$$\frac{g_1 = \langle V_1, E_1 \rangle \quad g_2 = \langle V_2, E_2 \rangle \quad g_3 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle}{\bar{g_1} \oplus \bar{g_2} \to \bar{g_3}}$$

$$\frac{g_1 = \langle V_1, E_1 \rangle \quad g_2 = \langle V_2, E_2 \rangle \quad g_3 = \langle V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2 \rangle}{\bar{g_1} \otimes \bar{g_2} \to \bar{g_3}}$$

FIGURE 1 – Sémantique à petits pas

$$[\![\mathsf{V}x]\!] \triangleq \langle \{x\}, \emptyset \rangle \tag{1}$$

$$[\![\mathsf{Empty}]\!] \triangleq \langle \emptyset, \emptyset \rangle \tag{2}$$

$$[\![e_1 \oplus e_2]\!] \triangleq \langle V_1 \cup V_2, E_1 \cup E_2 \rangle \qquad \text{si } [\![e_1]\!] = \langle V_1, E_1 \rangle \wedge [\![e_2]\!] = \langle V_2, E_2 \rangle \tag{3}$$

$$[\![e_1 \otimes e_2]\!] \triangleq \langle V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2 \rangle \qquad \text{si } [\![e_1]\!] = \langle V_1, E_1 \rangle \wedge [\![e_2]\!] = \langle V_2, E_2 \rangle \tag{4}$$

Pour les expressions étendues, on ajoute la règle suivante

$$[\![\bar{g}]\!] \triangleq g \tag{5}$$

FIGURE 2 – Sémantique dénotationnelle

(d) Assuming the termination of the new system, deduce the existence of a *unique* normal form for expressions.

After this question, we only study the deterministic semantics defined at the start.

5. State the theorems of correctness and adequacy of the two semantics on the graphs.

6. Prove correction.

7. Prove adequacy.

8. Prove the following equivalence in operational semantics. (Operational semantics has the same normal forms when it is not deterministic.)

$$\forall g, x \otimes (y \oplus z) \to^* \bar{g} \iff (x \otimes y) \oplus (x \otimes z) \to^* \bar{g}$$

9. Define a function $\mathsf{map} : (X \to Y) \times \mathsf{Expr}_X \to \mathsf{Expr}_Y$.

10. Calculate the denotational semantics of map on graphs.

11. What is the set of graphs constructed from the expressions Expr_X ?

12. Assuming the existence of a function $V : \mathsf{Expr}_X \to \mathcal{P}_f(X)$ which to a graph expression associates the set of its vertices, describe a function $N_x : \mathsf{Expr}_X \to \mathcal{P}_f(X)$ which to a graph expression e representing a graph $\langle V, E \rangle$ associates the set $\{y \in X \mid (x, y) \in E\}$. Justify that for all expressions $e \in \mathsf{Expr}_X$ such that $[\![e]\!] = \langle V, E \rangle$ we have the equality $N_x(e) = \{y \mid (x, y) \in E\}$.

Solution:

(Sketch)

1. This is what the progress theorem says.

$$\forall e \in \mathsf{Expr}_X^+, e \to e' \ \lor \ \exists g, e = \bar{g} \tag{6}$$

This is proved by a case analysis on the expression, we do not even require induction.

2. This is what the determinism theorem says.

$$\forall e, \forall e', \forall e'', e \to e' \land e \to e'' \implies e' = e'' \tag{7}$$

This is proved by induction on the expression e. **Note** we must take into account the fact that in the intermediate expressions $\bar{g}$ where g is a graph may appear. The base cases are evident. For the heredity, there is an evident case, which is $F[e]$. For the others, it is sufficient to see that g_3 is uniquely determined from g_1 and g_2.

3. This is what the termination theorem says.

$$\forall e, \sup\{k \mid \exists e', e \to^k e'\} < \infty \tag{8}$$

It suffices to show, for example, that $e \to e' \implies |e| > |e'|$ where $|e'| = 0$ if $e' = \bar{g}$ and 1 if it is a leaf expression, sum+1 when it is a binary operator.

4. We no longer require to have completely evaluated the expressions on the left to evaluate on the right... So there will be non-determinism.

   (a) Just consider the following expression

   $$\mathsf{V}x \oplus \mathsf{V}y \to \bar{g} \oplus \mathsf{V}y$$
   $$\mathsf{V}x \oplus \mathsf{V}y \to \mathsf{V}x \oplus \bar{g}$$

(b) A confluence theorem has the following form

$$\forall e, \forall e_1, \forall e_2, e \to e_1 \wedge e \to e_2 \implies \exists e_{1 \wedge 2}, e_1 \to^* e_{1 \wedge 2} \wedge e_2 \to^* e_{1 \wedge 2} \quad (9)$$

(c) To demonstrate this, we do an induction on the expression e, then a case analysis on the rule. **The only important rule is that on $F[e]$!**

(d) As the system is terminating, and using the progress theorem, there is a unique graph g such that $e \to^* g \not\to$. We will then want to say that the *sense* of the expression e is the unique graph to which it is reduced.

5. A correction theorem is written

$$\forall e, e \to^* \bar{g} \implies [\![e]\!] = g \quad (10)$$

An adequacy theorem is written

$$\forall e, [\![e]\!] = g \implies e \to^* \bar{g} \quad (11)$$

6. To show the correction, it suffices to show that all the rules are admissible in the given semantics. In other words, $[\![e]\!] = g \wedge e \to e' \implies [\![e']\!] = g$.

7. To show the adequacy, we proceed by induction on the expression e. We need the following lemma :

$$\forall e, \forall e', \forall F. \quad e \to^* e' \implies F[e] \to^* F[e']$$

8. We check it on expressions of type $\bar{g}$ because we come back to it by using non-deterministic reductions.

9. $\mathsf{map}\, f\, (e \oplus e') = (\mathsf{map}\, f\, e) \oplus (\mathsf{map}\, f\, e')$ etc ... We also need to show $\mathsf{map}\, f\, (\mathsf{V}x) = \mathsf{V}(fx)$.

10. The function $[\![\mathsf{map}]\!]$ is simply the function which to $\langle V, E \rangle$ associates $\langle f(V), f(E) \rangle$ where $f(E) \triangleq \{(f(u), f(v)) \mid (u, v) \in E\}$.

11. Any graph $g = \langle V, E \rangle$ is representable in the expressions Expr_X where $X \subseteq V$. It suffices to see that we can calculate

$$\bigoplus_{(u,v) \in E} (\mathsf{V}u) \otimes (\mathsf{V}v) \quad (12)$$

12. We define the function N as follows :

$$N_x(\mathsf{V}y) \triangleq \emptyset$$
$$N_x(\mathsf{Empty}) \triangleq \emptyset$$
$$N_x(e_1 \oplus e_2) \triangleq N(e_1) \cup N(e_2)$$
$$N_x(e_1 \otimes e_2) \triangleq N(e_1) \cup N(e_2) \cup \begin{cases} V(e_2) & \text{if } x \in V(e_1) \\ \emptyset & \text{otherwise} \end{cases}$$

Then we show by induction on e that $N_x(e)$ matches the neighbours x in $[\![e]\!]$. (Note : The interesting case is $\otimes$).

2   DCPOs

Rappel sur les familles dirigées

Une famille D non vide d'un ensemble $(X, \leq)$ est dirigée si et seulement si

$$\forall (x, y) \in D, \exists z \in D, z \geq x \wedge z \geq y$$

Rappels sur les DCPOs

Un DCPO est un ensemble partiellement ordonné $(X, \leq)$ tel que toute famille dirigée possède un sup. Un DCPO est *pointé* s'il existe un élément minimal.

Exercise 2 : Cartesian Closed Category

Show that the category of DCPOs is Cartesian closed, by going through the following steps :

1. Show that there exists a DCPO **1** such that for any DCPO D there exists a unique function continuous from **1** to D.

2. Show that if D_1 and D_2 are two DCPOs then $D_1 \times D_2$ with the product ordering is a DCPO.

3. Show that $D_1 \times D_2$ verifies a universal product property (where all the quantifications are on continuous functions).

$$\forall f : A \to D_1, g : A \to D_2, \exists! h : A \to D_1 \times D_2, \pi_1 \circ h = f \wedge \pi_1 \circ h = g$$

4. Show that $A \implies B$ the set of continuous functions from A to B ordered point to point is a DCPO.

5. Show that if A, B, C are DCPOs, then any continuous function $f : A \times B \to C$ transforms into a function $\Gamma f : A \to (B \implies C)$ which is also continuous.

6. Show that a function $f : A \times B \to C$ is continuous if and only if it is continuous in its two arguments.

7. Show that the evaluation map $\Delta : A \times (A \implies B) \to B$ is continuous.

Solution:

1. The DCPO is the empty set.

2. Consider a directed family of the product $F \subseteq D_1 \times D_2$. Let us show that $\sup F = (\sup \pi_1 F, \sup \pi_2 F)$.

   (a) The family $\pi_1 F$ is directed because π_1 is increasing. The same applies to $\pi_2 F$, so we have the existence of sups written above.

   (b) It is clear that for all $(x, y) \in F$, we have $x \leq \sup \pi_1 F$ and $y \leq \sup \pi_2 F$. So this pair is indeed an upper bound.

   (c) We consider (u, v) upper bound of F, clearly $u \geq \sup \pi_1 F$ and $v \geq \sup \pi_2 F$, so $(u, v) \geq (\sup \pi_1 F, \sup \pi_2 F)$. It is therefore the least upper bound.

3. Consider two functions f et g as described in the statement. We set $h(x) \triangleq (f(x), g(x))$. By construction we have the desired equation.

   (a) The function h is unique, indeed the equation determines the projections and therefore the pair.

   (b) The function h is monotonic, indeed $x \leq y$ implies $f(x) \leq f(y)$ and $g(x) \leq g(y)$ so $h(x) \leq h(y)$. It preserves the sup because it is calculated component by component.

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4. Consider a directed family of functions F from A to B. We show that $\sup F \triangleq x \mapsto \sup\{f(x) \mid f \in F\}$.

   (a) As the family F is directed, we deduce that for a fixed x, the family of $\{f(x) \mid f \in F\}$ is directed. So the function is well defined.

   (b) This function is point-to-point superior to all functions of F, and is the smallest point-to-point satisfying this property.

   (c) The function is monotonic, because $x \le y$ implies $f(x) \le f(y)$ for all $f \in F$, so the sup are comparable.

   (d) The function preserves the sup, because if G is a directed family of points

$$\sup_{x \in G} \sup_{f \in F} f(x) = \sup_{f \in F} \sup_{x \in G} f(x) = \sup_{f \in F} f(\sup_{x \in G} x) \tag{13}$$

5. We consider a function $f : A \times B \to C$. Let us suppose $\Gamma f : A \to (B \implies C)$ be the function defined by $\Gamma f : x \mapsto (y \mapsto f(x, y))$.

   (a) For all $x \in A$, the function $y \mapsto f(x, y)$ is Scott-continuous.

   (b) The function Γf is monotonic (evident).

   (c) The function Γf preserves the sup. Let G be a directed family of A. By construction, we have the following equality

$$\sup_{x \in G} (\Gamma f)(x) \triangleq y \mapsto \sup_{x \in G} f(x, y) \tag{14}$$

   So we have

$$\sup_{x \in G} (\Gamma f)(x) = y \mapsto f(\sup G, y) = (\Gamma f)(\sup G) \tag{15}$$

6. Let f be a continuous function, then by fixing an argument it remains trivially continuous. Conversely, suppose f is continues in each of its arguments.

   (a) The function f is monotonic, because monotonous in its two arguments, by construction of the product.

   (b) The function preserves the sup because

$$
\begin{aligned}
f\left(\sup_{(a,b) \in F} (a, b)\right) &= f\left(\sup \pi_1 F, \sup \pi_2 F\right) \\
&= \sup_{x \in \pi_1 F} f\left(x, \sup \pi_2 F\right) \\
&= \sup_{x \in \pi_1 F} \sup_{y \in \pi_2 F} f(x, y) \\
&= \sup_{(x,y) \in F} f(x, y)
\end{aligned}
$$

   Note that the last equality only holds because the family is *directed.*

7. We have the following two steps :

   (a) Let us show that the evaluation function is increasing. Let $(x, f) \le (y, g)$.

$$f(x) \le g(x) \le g(y) \tag{16}$$

   (b) Let us show that the function preserves the sup. Using the previous question, we only need to check by fixing f or by fixing x, which renders the proof trivial.

3   Topology

Topologie

Une topologie τ sur un ensemble X est un ensemble de parties de X qui vérifie

1. τ est stable par intersection finie.
2. τ est stable par union quelconque.
3. τ contient l'ensemble X.
4. τ contient l'ensemble $\emptyset$.

On dira alors d'un élément de τ qu'il est *ouvert*. Le complémentaire d'un ouvert est par définition un ensemble *fermé*.

Fonction continue

Une fonction $f : X \to Y$ est *continue* de (X, τ) vers (Y, θ) si et seulement si

$$\forall U \in \theta, f^{-1}(U) \in \tau$$

Topologie de Scott

Soit $(D, \leq)$ un DCPO. Une partie $U \subseteq D$ est appellée un *ouvert de Scott* si et seulement si elle vérifie

1. U est clos vers le haut :

$$\forall x, \forall y. \quad x \in U \wedge x \leq y \implies y \in U$$

2. U est inaccessible par le bas :

$$\forall E \text{ dirigée} \quad \sup E \in U \implies E \cap U \neq \emptyset$$

Exercise 3 : Scott topology

1. Show that the Scott toplogy is a topology.
2. Show that a closed set of D is closed at the bottom and closed under suprema of directed subsets.
3. Show that $\downarrow x \triangleq \{y \in D \mid y \leq x\}$ is a closed set of D for Scott topology.
4. Show that the continuous functions for Scott topology are the Scott-continuous functions.

Solution:

1. Stability by arbitrary union is trivial. Finite intersection is done just by noticing that we take an element above all the elements that are in each of the U_i.

2. The complement of a closed set by the top is closed by the bottom, and it is easy to see that inaccessible from below means that its complementary contains the sup.

3. Its complement is open : clearly it is closed by the top, and inaccessible from below.

4. (a) Let us suppose f Scott-continuous, we show that $f^{-1}(U)$ is open when U is an open set. As f is monotonic, we have $f^{-1}(U)$ closed from the top. Let us suppose $\sup F \in f^{-1}(U)$, then $f(\sup F) \in U$, so $\sup f(F) \in U$ so $f(F) \cap U \neq \emptyset$ so $F \cap f^{-1}(U) \neq \emptyset$.

(b) Reciprocally, we consider f continuous. We show that f is monotonic. If $x \leq y$, we consider $F = \downarrow f(y)$ which is closed. We have $y \in f^{-1}(F)$ and as it is closed at the bottom, x also, therefore $f(x) \in F$, therefore $f(x) \leq f(y)$.

Let us now show that f preserves the sup. Let F be a directed family, we already know since f is monotonic that

$$f(\sup x) \geq \sup f(x) \tag{17}$$

It remains to show that $f(\sup F) \leq \sup f(x)$. For this, we consider $G = \downarrow \sup f(F)$ which is a closed set. Hence, we have $f^{-1}(G)$ which is also closed. By construction, $F \subseteq f^{-1}(G)$. Since it is a closed set, it is stable by sup of the directed family therefore $\sup F \in f^{-1}(G)$, this proves

$$f(\sup F) \leq \sup f(F) \tag{18}$$

And we conclude.

Exercise 4 : Finite words ... or infinite

Let $S = \{0,1\}^\infty = \{0,1\}^* \cup \{0,1\}^\omega$, with the prefix-ordering.

1. Show that S is a DCPO. Is it a lattice ?

2. What are the maximal elements of S ?

3. Let f be a function from S to $\mathbf{Bool}_\perp$ such that :

$$\forall s \in \{0,1\}^\omega \begin{cases} f(s) = 1 & \text{if } s \text{ contains the factor } 0 \cdot 1 \\ f(s) = 0 & \text{otherwise} \end{cases}$$

Show that f is not Scott-continuous.

4. We consider the function $v : S \to J$ defined by :

$$v(b_1 \cdot b_2 \cdots b_n) = \left[\sum_{i=1}^n 2^{-i} b_i, \sum_{i=1}^n 2^{-i} b_i + 2^{-n} \right]$$

$$v(b_1 \cdot b_2 \cdots) = \left\{ \sum_{i=1}^\infty 2^{-i} b_i \right\}$$

What is v for ? Show that v is Scott-continuous. Is it injective ?

5. Let g be a Scott-continuous function from S to $\mathbf{Bool}_\perp$ which is compatible with v :

$$\forall x, y \in S, v(x) = v(y) \to g(x) = g(y)$$

Show that if $\forall x \in \{0,1\}^\omega$, $g(x) \neq \perp$, then g is constant over $\{0,1\}^\infty$.

Solution:

(Sketch)

1. Consider a directed family F for the prefix order. Either it is stationary and its sup is trivial, or the sup of the size is infinite and then we can construct an infinite word that matches. Note that this is **not** an inf-semilattice : indeed, the empty set has no lower bound.

2. The maximum elements of S are the infinite words.

3. If the function f was Scott-continuous, then $f^{-1}(\{0\})$ is Scott-open. Then, when we consider the sequence 0^k there exists a finite k for which $f(0^k) = 0$ but so $f(0^k \cdot 1) = 0$ by monotonicity, which is a contradiction.

4. The function v allows to interpret the infinite words in the variable precision reals in $[0, 1]$. The function is clearly monotonous because of 2^n which consumes the rest. The sums being positive real, everything commutes wrt sup.

   Note that the function is not injective, because the word 01^ω is equal to the word 1.

5. If g is not constant, we can construct $\hat{g}$ which corresponds to lifting g according to v, and which is not constant anymore. From the previous exercise, this is a contradiction.