Programmation 1

TD n°7

3 novembre 2020

Exercise 1 :

1. Give a proof of

$$((x \dot{+} (\dot{-}y)) \dot{+} \dot{2}, \rho[x \mapsto 3, y \mapsto 2]) \to_{pp}^{*} (\dot{3}, \rho[x \mapsto 3, y \mapsto 2])$$

2. State then prove the progress theorem (théorème de progrès)
3. State then prove the determinism theorem (théorème de déterminisme)
4. Show the correctness of denotational semantics
5. Show the adequacy of denotational semantics

Solution:

1. We give a derivation tree in small steps, noting $\rho' = \rho[x \mapsto 3, y \mapsto 2]$:

$$\dfrac{\dfrac{\dfrac{}{(x,\rho) \to_{pp} (\dot{3},\rho)}\;(\texttt{Var})}{(x \dot{+} (\dot{-}y), \rho') \to_{pp} (\dot{3} \dot{+} (\dot{-}y), \rho')}\;(+_\ell)}{((x \dot{+} (\dot{-}y)) \dot{+} \dot{2}, \rho') \to_{pp} ((\dot{3} \dot{+} (\dot{-}y)) \dot{+} \dot{2}, \rho')}\;(+_\ell)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(y,\rho') \to_{pp} (\dot{2},\rho')}\;(\texttt{Var})}{(\dot{-}y, \rho) \to_{pp} (\dot{-}\dot{2}, \rho)}\;(-)}{(\dot{3} \dot{+} (\dot{-}y), \rho') \to_{pp} (\dot{3} \dot{+} (\dot{-}\dot{2}), \rho')}\;(+_r)}{((\dot{3} \dot{+} (\dot{-}y)) \dot{+} \dot{2}, \rho') \to_{pp} ((\dot{3} \dot{+} (\dot{-}\dot{2})) \dot{+} \dot{2}, \rho')}\;(+_\ell)$$

$$\dfrac{\dfrac{\dfrac{}{(\dot{-}\dot{2}, \rho) \to_{pp} (\dot{\widehat{-2}}, \rho)}\;(-_{\text{fin}})}{(\dot{3} \dot{+} (\dot{-}\dot{2}), \rho') \to_{pp} (\dot{3} \dot{+} (\dot{\widehat{-2}}), \rho')}\;(+_r)}{((\dot{3} \dot{+} (\dot{-}\dot{2})), \dot{+}\dot{2}, \rho') \to_{pp} ((\dot{3} \dot{+} (\dot{\widehat{-2}})) \dot{+} \dot{2}, \rho')}\;(+_\ell)$$

$$\dfrac{\dfrac{}{(\dot{3} \dot{+} \dot{\widehat{-2}}, \rho') \to_{pp} (\dot{1}, \rho')}\;(+_{\text{fin}})}{((\dot{3} \dot{+} \dot{\widehat{-2}}) \dot{+} \dot{2}, \rho') \to_{pp} (\dot{1} \dot{+} \dot{2}, \rho')}\;(+_\ell)$$

$$\dfrac{}{(\dot{1} \dot{+} \dot{2}, \rho') \to_{pp} (\dot{3}, \rho')}\;(+_{\text{fin}})$$

We obtain the derivation :

$$((x \dotplus (\dot{-}y)) \dotplus \dot{2}, \rho[x \mapsto 3, y \mapsto 2]) \rightarrow^*_{pp} (\dot{3}, \rho[x \mapsto 3, y \mapsto 2])$$

2. *(Progress) The only configurations which do not have a successor by the relation $\rightarrow_{pp}$ are configurations of the form $(\dot{n}, \rho)$.*
Proof Let ρ be an environment. We proceed by structural induction over the expression e.
Base case $e = \dot{n}$: The configuration (e, ρ) does not have a successor.
Base case $e = x$: We have the derivation :

$$\frac{}{(x, \rho) \rightarrow_{pp} (\dot{\widehat{\rho(x)}}, \rho)} \ (\texttt{Var})$$

Therefore, we have a derivation $(e, \rho) \rightarrow_{pp} (\dot{\widehat{\rho(x)}}, \rho)$ and the configuration (e, ρ) admits a successor.
Case $e = \dot{-}e_0$: We proceed by a disjunction of cases over the form of the expression e_0.
If $e_0 = \dot{n}$: We have the derivation

$$\frac{}{(\dot{-}\dot{n}, \rho) \rightarrow_{pp} (\dot{\widehat{-n}}, \rho)} \ (-_{\text{fin}})$$

Hence, the configuration (e, ρ) admits a successor.
If not, $\forall n,\ e_0 \neq \dot{n}$: By the induction hypothesis, the configuration (e_0, ρ) admits a successor for $\rightarrow_{pp}$. Let it be (e'_0, ρ). We have the derivation :

$$\frac{(e_0, \rho) \rightarrow_{pp} (e'_0, \rho)}{(\dot{-}e_0, \rho) \rightarrow_{pp} (\dot{-}e'_0, \rho)} \ (-)$$

Therefore, the configuration (e, ρ) admits a successor.
Case $e = e_1 \dotplus e_2$: We proceed by a disjunction of cases over the form of the expressions $e1$ and e_2.
Case $e_1 = \dot{n}$ and $e_2 = \dot{m}$. We have the derivation :

$$\frac{}{(\dot{n} \dotplus \dot{m}, \rho) \rightarrow_{pp} (\dot{\widehat{n + m}}, \rho)} \ (+_{\text{fin}})$$

Therefore, the configuration (e, ρ) admits a successor.
Case $e_1 = \dot{n}$ and $\forall m,\ e_2 \neq \dot{m}$: By induction hypothesis, the configuration (e_2, ρ) admits a successor for $\rightarrow_{pp}$. We denote it as (e'_2, ρ). We have the derivation :

$$\frac{(e_2, \rho) \rightarrow_{pp} (e'_2, \rho)}{(\dot{n} \dotplus e_2, \rho) \rightarrow_{pp} (\dot{n} \dotplus e'_2, \rho)} \ (+_r)$$

Therefore, the configuration (e, ρ) admits a successor.
Case $\forall n,\ e_1 \neq \dot{n}$: By induction hypothesis, the configuration (e_1, ρ) admits a successor for $\rightarrow_{pp}$. Let us denote it by (e'_1, ρ). We have the derivation :

$$\frac{(e_1, \rho) \rightarrow_{pp} (e'_1, \rho)}{(e_1 \dotplus e_2, \rho) \rightarrow_{pp} (e'_1 \dotplus e_2, \rho)} \ (+_\ell)$$

Therefore, the configuration (e, ρ) admits a successor.
By the principle of induction, the only configurations that do not admit a successor for $\rightarrow_{pp}$ are the configurations $(\dot{n}, \rho)$.

3. *(Determinism) The reduction $\rightarrow_{pp}$ is deterministic, i.e. for all* e, e_1, e_2, ρ*, if* $(e,\rho) \rightarrow_{pp} (e_1,\rho)$ *and* $(e,\rho) \rightarrow_{pp} (e_2,\rho)$*, then* $e_1 = e_2$*.*

   Proof Let ρ be an environment. We proceed by structural induction on the expression e. We assume that there exists e_1 and e_2 such that $(e,\rho) \rightarrow_{pp} (e_1,\rho)$ and $(e,\rho) \rightarrow_{pp} (e_2,\rho)$.

   Base case $e = \dot{n}$: There does not exist any derivation rule for the configuration (e,ρ), hence, it has no successor : It holds vacuously.

   Base case $e = x$: The only rule that is applicable at the configuration (e,ρ) is :

$$\frac{}{(x,\rho) \rightarrow_{pp} (\widehat{\dot{\rho(x)}},\rho)}\ (\texttt{Var})$$

   Therefore, $e_1 = e_2 = \widehat{\dot{\rho(x)}}$.

   Case $e = \dot{-}e_0$: We proceed by a disjunction of cases of the form of the expression e_0.

   If $e_0 = \dot{n}$: The only rule applicable at the configuration (e,ρ) is :

$$\frac{}{(\dot{-}\dot{n},\rho) \rightarrow_{pp} (\widehat{\dot{-n}},\rho)}\ (-_{\text{fin}})$$

   In effect, by progress, the configuration (e_0,ρ) cannot be reduced, which renders the rule $(-)$ inapplicable. Therefore, $e_1 = e_2 = \widehat{\dot{-n}}$.

   Otherwise, $\forall n,\ e_0 \neq \dot{n}$: The only rule applicable at the configuration (e,ρ) is $(-)$, therefore, there exists e_1' and e_2' such that $e_1 = \dot{-}e_1'$, $e_2 = \dot{-}e_2'$, and so :

$$\frac{(e_0,\rho) \rightarrow_{pp} (e_1',\rho)}{(\dot{-}e_0,\rho) \rightarrow_{pp} (\dot{-}e_1',\rho)}\ (-)$$

   and

$$\frac{(e_0,\rho) \rightarrow_{pp} (e_2',\rho)}{(\dot{-}e_0,\rho) \rightarrow_{pp} (\dot{-}e_2',\rho)}\ (-)$$

   By the induction hypothesis on e_0, $e_1' = e_2'$ and consequently, $e_1 = e_2$.

   Case $e = e_3 \dot{+} e_4$: We proceed by a case disjunction over the forms of expressions e_3 and e_4.

   Case $e_3 = \dot{n}$, $e_4 = \dot{m}$: The only rule applicable at the configuration (e,ρ) is :

$$\frac{}{(\dot{n} \dot{+} \dot{m},\rho) \rightarrow_{pp} (\widehat{\dot{n+m}},\rho)}\ (+_{\text{fin}})$$

   In effect, by progress, the configurations (e_3,ρ) and (e_4,ρ) can't be reduced, which renders the rules $(+_\ell)$ and $(+_r)$ inapplicable. Therefore, $e_1 = e_2 = \widehat{\dot{n+m}}$.

   Case $e_3 = \dot{n}$, $\forall m,\ e_4 \neq \dot{m}$: : The only rule applicable at the configuration (e,ρ) is $(+_r)$, hence, there exists e_1' and e_2' such that $e_1 = \dot{n} \dot{+} e_1'$, $e_2 = \dot{n} \dot{+} e_2'$, and so :

$$\frac{(e_4,\rho) \rightarrow_{pp} (e_1',\rho)}{(\dot{n} \dot{+} e_4,\rho) \rightarrow_{pp} (\dot{n} \dot{+} e_1',\rho)}\ (+_r)$$

   and

$$\frac{(e_4, \rho) \rightarrow_{pp} (e_2', \rho)}{(\dot{n} \dot{+} e_4, \rho) \rightarrow_{pp} (\dot{n} \dot{+} e_2', \rho)} \ (+_r)$$

In effect, by progress, the configuration (e_3, ρ) cannot be reduced, which renders the $(+_\ell)$ rule inapplicable. By induction hypothesis on e_4, $e_1' = e_2'$. We can then deduce $e_1 = e_2$.

Case $\forall n,\, e_3 \neq \dot{n}$: The only rule applicable at the configuration (e, ρ) is $(+_\ell)$, hence, there exists e_1' and e_2' such that $e_1 = e_1' \dot{+} e_4$, $e_2 = e_2' \dot{+} e_4$, and so :

$$\frac{(e_3, \rho) \rightarrow_{pp} (e_1', \rho)}{(e_3 \dot{+} e_4, \rho) \rightarrow_{pp} (e_3 \dot{+} e_4, \rho)} \ (+_\ell)$$

and

$$\frac{(e_3, \rho) \rightarrow_{pp} (e_2', \rho)}{(e_3 \dot{+} e_4, \rho) \rightarrow_{pp} (e_3 \dot{+} e_4, \rho)} \ (+_\ell)$$

By induction hypothesis on e_3, $e_1' = e_2'$. We deduce that $e_1 = e_2$.

By the principle of induction, the reduction is deterministic.

4. *(Correction) Let $n \in \mathbb{N}$, ρ be an environment, and e an expression. If $[\![e]\!]_\rho = n$, then there exists a derivation $(e, \rho) \rightarrow_{pp}^* (\dot{n}, \rho)$.*

Proof Let ρ be an environment and e an expression. We proceed by structural induction on the expression e. Let us assume there exists an integer $n \in \mathbb{N}$ such that $[\![e]\!]_\rho = n$.

Base case $e = \dot{m}$: By definition of denotational semantics, $[\![e]\!]_\rho = m$, hence, $n = m$. Furthermore, $(e, \rho) \rightarrow_{pp}^0 (\dot{m}, \rho) = (\dot{n}, \rho)$.

Base case $e = x$: By definition of denotational semantics, $[\![e]\!]_\rho = \rho(x)$, hence, $n = \rho(x)$. Furthermore, $(e, \rho) \rightarrow_{pp} (\dot{\widehat{\rho(x)}}, \rho) = (\dot{n}, \rho)$ by the rule (`Var`).

Case $e = \dot{-} e_0$: We denote $m = [\![e_0]\!]_\rho$. By the definition of denotational semantics, $[\![e]\!]_\rho = -[\![e_0]\!]_\rho = -m$, hence $n = -m$. By the induction hypothesis on e_0, we have that $(e_0, \rho) \rightarrow_{pp}^* (\dot{m}, \rho)$. We need an intermediate lemma here because we cannot use this sequence of reduction of arbitrary length in the derivation rules. We need to decompose this reduction into individual steps.

Lemma 1 If $(e_0, \rho) \rightarrow_{pp}^* (\dot{m}, \rho)$, then $(\dot{-} e_0, \rho) \rightarrow_{pp}^* (\dot{\widehat{-m}}, \rho)$.

Proof We proceed by induction on the size of the reduction.

— If $(e_0, \rho) \rightarrow_{pp}^0 (\dot{m}, \rho)$, then $e_0 = \dot{m}$. We have then that $(\dot{-} e_0, \rho) \rightarrow_{pp} (\dot{\widehat{-m}}, \rho) = (\dot{n}, \rho)$ by the rule $(-_{\text{fin}})$.

— If $(e_0, \rho) \rightarrow_{pp}^{k+1} (\dot{m}, \rho)$, then there exists e_1 such that $(e_0, \rho) \rightarrow_{pp} (e_1, \rho)$ and $(e_1, \rho) \rightarrow_{pp}^k (\dot{m}, \rho)$. By induction hypothesis, $(\dot{-} e_1, \rho) \rightarrow_{pp}^* (\dot{\widehat{-m}}, \rho)$. Furthermore,

$$\frac{(e_0, \rho) \rightarrow_{pp} (e_1, \rho)}{(\dot{-} e_0, \rho) \rightarrow_{pp} (\dot{-} e_1, \rho)} \ (-)$$

We conclude that $(\dot{-} e_0, \rho) \rightarrow_{pp}^* (\dot{\widehat{-m}}, \rho)$.

By the lemma, $(e, \rho) \rightarrow_{pp}^* (\dot{\widehat{-m}}, \rho) = (\dot{n}, \rho)$.

Case $e = e_1 \dot{+} e_2$: We proceed by case disjunction on the forms of the expression e_1.

Case $e_1 = \dot{m}_1$: We denote $m_2 = [\![e_2]\!]_\rho$. By the definition of denotational semantics, $[\![e]\!]_\rho = [\![e_1]\!]_\rho + [\![e_2]\!]_\rho = m_1 + m_2$, so $n = m_1 + m_2$. By induction hypothesis on e_2, we have that $(e_2, \rho) \to_{pp}^* (\dot{m}_2, \rho)$. Here, we need an intermediate lemma.

Lemma 2 If $(e_2, \rho) \to_{pp}^* (\dot{m}_2, \rho)$, then $(\dot{m}_1 \dotplus e_2, \rho) \to_{pp}^* (\widehat{m_1 + m_2}, \rho)$.

Proof We proceed by induction on the size of the reduction.

— If $(e_2, \rho) \to_{pp}^0 (\dot{m}_2, \rho)$, then $e_2 = \dot{m}_2$. We then have $(\dot{m}_1 \dotplus e_2, \rho) \to_{pp} (\widehat{m_1 + m_2}, \rho)$ $= (\dot{n}, \rho)$ by the rule $(+_{\text{fin}})$.

— If $(e_2, \rho) \to_{pp}^{k+1} (\dot{m}_2, \rho)$, then there exists e_3 such that $(e_2, \rho) \to_{pp} (e_3, \rho)$ and $(e_3, \rho) \to_{pp}^k (\dot{m}_2, \rho)$. By induction hypothesis, $(\dot{m}_1 \dotplus e_3, \rho) \to_{pp}^* (\widehat{m_1 + m_2}, \rho)$. Furthermore,

$$\frac{(e_2, \rho) \to_{pp} (e_3, \rho)}{(\dot{m}_1 \dotplus e_2, \rho) \to_{pp} (\dot{m}_1 \dotplus e_3, \rho)} \ (+_r)$$

We conclude that $(\dot{m}_1 \dotplus e_2, \rho) \to_{pp}^* (\widehat{m_1 + m_2}, \rho)$.

By the lemma, $(e, \rho) \to_{pp}^* (\widehat{m_1 + m_2}, \rho) = (\dot{n}, \rho)$.

Case $\forall n,\ e_1 \neq \dot{n}$: We denote $m_1 = [\![e_1]\!]_\rho$ and $m_2 = [\![e_2]\!]_\rho$. By the definition of denotational semantics, $[\![e]\!]_\rho = [\![e_1]\!]_\rho + [\![e_2]\!]_\rho = m_1 + m_2$, so $n = m_1 + m_2$. By the induction hypothesis on e_1 and e_2, we have that $(e_1, \rho) \to_{pp}^* (\dot{m}_1, \rho)$ and $(e_2, \rho) \to_{pp}^* (\dot{m}_2, \rho)$. We once again need an intermediate lemma here.

Lemma 3 If $(e_1, \rho) \to_{pp}^* (\dot{m}_1, \rho)$ then $(e_1 \dotplus e_2, \rho) \to_{pp}^* (\dot{m}_1 \dotplus e_2, \rho)$.

Proof We proceed by induction on the size of the reduction.

— If $(e_1, \rho) \to_{pp}^0 (\dot{m}_1, \rho)$, then $e_1 = \dot{m}_1$. We then have that $(e_1 \dotplus e_2, \rho) \to_{pp}^0 (\dot{m}_1 \dotplus e_2, \rho)$.

— If $(e_1, \rho) \to_{pp}^{k+1} (\dot{m}_1, \rho)$, then there exists e_3 such that $(e_1, \rho) \to_{pp} (e_3, \rho)$ and $(e_3, \rho) \to_{pp}^k (\dot{m}_1, \rho)$. By induction hypothesis, $(e_3 \dotplus e_2, \rho) \to_{pp}^* (\dot{m}_1 \dotplus e_2, \rho)$. Furthermore,

$$\frac{(e_1, \rho) \to_{pp} (e_3, \rho)}{(e_1 \dotplus e_2, \rho) \to_{pp} (e_3 \dotplus e_2, \rho)} \ (+_\ell)$$

We conclude that $(e_1 \dotplus e_2, \rho) \to_{pp}^* (\dot{m}_1 \dotplus e_2, \rho)$.

By the lemma, $(e, \rho) \to_{pp}^* (\dot{m}_1 \dotplus e_2, \rho)$. By the Lemma 2, we also have that $(\dot{m}_1 \dotplus e_2, \rho) \to_{pp}^* (\widehat{m_1 + m_2}, \rho) = (\dot{n}, \rho)$. We conclude by concatenating both the sequences of the reduction.

By the principle of induction, the denotational semantics is correct.

5. *(Adequacy) Let* $n \in \mathbb{N}$, ρ *be an environment, and* e *an expression. If there exists a derivation* $(e, \rho) \to_{pp}^* (\dot{n}, \rho)$, *then* $[\![e]\!]_\rho = n$.

   Proof We let ρ be an environment, and e an expression. We start by showing that the reduction $\to_{pp}$ preserves the denotational semantics, i.e. for all steps $(e, \rho) \to_{pp} (e', \rho)$, $[\![e']\!]_\rho = [\![e]\!]_\rho$. We proceed by structural induction on the expression e.

   Base case $e = \dot{m}$: The configuration (e, ρ) does not admit a successor - there is nothing to prove.

   Base case $e = x$: We have $(e, \rho) \to_{pp} (\widehat{\rho(x)}, \rho) = (\dot{n}, \rho)$ by the rule (`Var`), and it is the only possible step by determinism. It is therefore sufficient to verify that this step preserves the denotational semantics. By definition of denotational semantics, $[\![e]\!]_\rho = \rho(x) = [\![\widehat{\rho(x)}]\!]_\rho$.

Case $e = \dot{-}e_0$: We proceed by a case disjunction on the form of the expression e_0.

If $e_0 = \dot{n}$, we have :

$$\frac{}{(\dot{-}\dot{n},\rho) \rightarrow_{pp} (\widehat{\dot{-}n},\rho)}\ (-_{\text{fin}})$$

It is the only step possible, by determinism. Furthermore, by the definition of denotational semantics, $[\![e]\!]_\rho = -[\![e_0]\!]_\rho = -n = [\![\widehat{\dot{-}n}]\!]_\rho$.

Otherwise, $\forall n,\, e_0 \neq \dot{n}$: By progress, there exists e_1 such that $(e_0,\rho) \rightarrow_{pp} (e_1,\rho)$. We have :

$$\frac{(e_0,\rho) \rightarrow_{pp} (e_1,\rho)}{(\dot{-}e_0,\rho) \rightarrow_{pp} (\dot{-}e_1,\rho)}\ (-)$$

Therefore, $(e,\rho) \rightarrow_{pp} (\dot{-}e_1,\rho)$, and by determinism it is the only step possible. By induction hypothesis on e_0, $[\![e_0]\!]_\rho = [\![e_1]\!]_\rho$. Finally, by the definition of denotational semantics, $[\![e]\!]_\rho = -[\![e_0]\!]_\rho = -[\![e_1]\!]_\rho = [\![\dot{-}e_1]\!]_\rho$.

Case $e = e_1 \dot{+} e_2$: We proceed by a disjunction of cases on the forms of the expressions e_1 and e_2.

Case $e_1 = \dot{n}$ and $e_2 = \dot{m}$: We have the derivation :

$$\frac{}{(\dot{n} \dot{+} \dot{m},\rho) \rightarrow_{pp} (\widehat{\dot{n+m}},\rho)}\ (+_{\text{fin}})$$

By determinism, it is the only step possible. Moreover, by the definition of denotational semantics, $[\![e]\!]_\rho = [\![e_1]\!]_\rho + [\![e_2]\!]_\rho = n + m = [\![\widehat{\dot{n+m}}]\!]_\rho$.

Case $e_1 = \dot{n}$ and $\forall m,\, e_2 \neq \dot{m}$: By progress, there exists e_3 such that $(e_2,\rho) \rightarrow_{pp} (e_3,\rho)$. We have the derivation :

$$\frac{(e_2,\rho) \rightarrow_{pp} (e_3,\rho)}{(\dot{n} \dot{+} e_2,\rho) \rightarrow_{pp} (\dot{n} \dot{+} e_3,\rho)}\ (+_r)$$

By determinism, it is the only step possible. By induction hypothesis, $[\![e_2]\!]_\rho = [\![e_3]\!]_\rho$. Finally, by the definition of denotational semantics, $[\![e]\!]_\rho = [\![e_1]\!]_\rho + [\![e_2]\!]_\rho = n + [\![e_3]\!]_\rho = [\![\dot{n} \dot{+} e_3]\!]_\rho$.

Case $\forall n,\, e_1 \neq \dot{n}$: By progress, there exists e_3 such that $(e_1,\rho) \rightarrow_{pp} (e_3,\rho)$. We have the derivation :

$$\frac{(e_1,\rho) \rightarrow_{pp} (e_3,\rho)}{(e_1 \dot{+} e_2,\rho) \rightarrow_{pp} (e_3 \dot{+} e_2,\rho)}\ (+_\ell)$$

By determinism, it is the only step possible. By induction hypothesis, $[\![e_1]\!]_\rho = [\![e_3]\!]_\rho$. Finally, by the definition of denotational semantics, $[\![e]\!]_\rho = [\![e_1]\!]_\rho + [\![e_2]\!]_\rho = [\![e_3]\!]_\rho + [\![e_2]\!]_\rho = [\![e_3 \dot{+} e_2]\!]_\rho$.

By principle of induction, the reduction $\rightarrow_{pp}$ preserves denotational semantics. We deduce the adequacy by induction over the length of the derivation $(e,\rho) \rightarrow^*_{pp} (\dot{n},\rho)$.

- If $(e,\rho) \rightarrow^0_{pp} (\dot{n},\rho)$, $e = \dot{n}$ and $[\![e]\!]_\rho = n$.
- If $(e,\rho) \rightarrow^{k+1}_{pp} (\dot{n},\rho)$, there exists an expression e' such that $(e,\rho) \rightarrow_{pp} (e',\rho)$ and $(e',\rho) \rightarrow^k_{pp} (\dot{n},\rho)$. By induction hypothesis, $[\![e']\!]_\rho = n$. The reduction $\rightarrow_{pp}$ preserves denotation semantics, $[\![e]\!]_\rho = [\![e']\!]_\rho$. We deduce that $[\![e]\!]_\rho = n$.

By the principle of induction, the denotational semantics is adequate.