Tree Automata and their Applications

TD n°4 : Extensions and PDL

2021-2022

Exercise : Alternating Word Automata

Definition 1 *If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi$.*

Definition 2 *A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subset Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word w is a tree t labelled by Q such that :*
— *if $w = \varepsilon$, then $t = q_0$ with $q_0 \in Q_0$.*
— *if $w = a.w'$, then $t = q_0(t_1, \ldots, t_n)$ $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, q_i, Q_f, \delta)$ and $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.*

Definition 3 *We say that a run is accepting if every leaf of the form q satisfies that $q \in Q_f$.*

1. Show how to reduce the emptiness problem for an AWA on a one letter alphabet $\{a\}$ with formulas that are in positive disjunctive normal form to the emptiness problem of a tree automaton.
2. Show how to reduce the emptiness problem for a tree automaton to the emptiness problem of an AWA on a one letter alphabet $\{a\}$. Conclude on the complexity of the emptiness problem for an AWA on a one letter alphabet.

Solution:

1. Let $\mathcal{A} = (Q, \{a\}, q_0, Q_f, \delta)$ an AWA. Notice that δ only contains one rule. The accepted trees then have a very particular form, which we can recognize using an NFTA. We construct an NFTA of the form
$(Q, \{f_k(k) \mid 0 \leq k \leq n\}, F, \Delta')$ with $F = Q_0$:

$$\delta(q, a) = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{k_i} (q_{i,j}, i) \Rightarrow \forall i, f_i(q_{i,1}, ..., q_{i,k_i}) \longrightarrow q \in \delta'$$

2. Let $\mathcal{A} = (Q, F, \delta, Q_f)$ be an NFTA.
We build an instance $\langle \mathcal{A}', a^{|Q|} \rangle$ of the membership problem in AWA, where $\mathcal{A}' := (Q', \sigma, \delta', q_0)$, where $Q' := Q \times \mathcal{F} \uplus \{q_0\}$. For a pair (q, f) in $Q \times \mathcal{F}_n$ with $n \geq 0$, we define the formula

$$\delta'((q, f), a) := \bigvee_{(q, f, q_1, \ldots, q_n) \in \delta} \bigwedge_{i \in \{1, \ldots, n\}} \bigvee_{g \in \mathcal{F}} (q_i, g)$$

Regarding the initial state q_0, we make a disjunction over all possible pairs in $Q_f \times \mathcal{F}$.

$$\delta'(q_0, a) := \bigvee_{q \in Q_f} \bigvee_{f \in \mathcal{F}} \delta'((q, f), a)$$

Now, we prove that our construction is correct.

If $L(\mathcal{A}) \neq \emptyset$, then there exists a tree of height at most $|Q|$ in $L(\mathcal{A})$ (by the pigeonhole principle).

Let us assume that $L(\mathcal{A})$ is not empty. Then there exists a tree $t \in L(\mathcal{A})$ with height at most $|Q|$, i.e. $|p| < |Q|$ for all $p \in \texttt{dom}\ t$. Thus there exists an accepting run $\rho \in T(Q)$ with $\texttt{dom}\ \rho = \texttt{dom}\ t$, $\rho(\varepsilon) \in Q_f$, and every elementary tree in ρ is consistent with δ. We can map the pair t, ρ to a tree t' in $T(Q \times \{0, \dots, |Q|?1\})$ with domain $\texttt{dom}\ t' := \texttt{dom}\ t$.

It remains to show that t' is an accepting run of $\mathcal{A}'$ on $a^{|Q|}$. Let p be a position in $\texttt{dom}\ t'$ other than the root. Then $t(p) = f$ and $\rho(p) = q$ for some $f \in \mathcal{F}_n$ for some n and $q \in Q$. Since ρ is a run of $\mathcal{A}$, p has n children $p1, \dots, pn$ and there exists a transition $(q, f, q_1, \dots, q_n) \in \delta$ with $\rho(pi) = q_i$ for all $1 \leq i \leq n$; let also $t(pi) = g_i \in \mathcal{F}$ for all $1 \leq i \leq n$. Finally, for the root, there exists $\rho(\varepsilon) \in Q_f$ and $t(\varepsilon)?\mathcal{F}_n$ for some n such that, in a similar way, the children satisfy the requirement.

Conversely, assume there is an accepting run $t' \in T(Q)$ of $\mathcal{A}'$.

First, we show that every non-root position p of t' such that $t'(p) = q$ has exactly n children for some $(q, f, q_1, \dots, q_n) \in \delta$. Indeed, the list of children must satisfy $q_1 \wedge \cdots \wedge q_n$ for some $(q, f, q_1, \dots, q_n)$. Hence, this list must contain at least one occurence of each state q_i. If there exists a child position pi such that $t(pi) \notin \{q_1, \dots, q_n\}$, then the entire subtree can be removed, and we still have an accepting tree. Conversely, if p is smaller, there is a child position pi with $t'(pi) = q$, and that can be duplicated as many times as needed. Finally, the children can be reordered such that $t'(pi) = q_i$ since the ordering does not matter for acceptance. We can extend this assumption for the root.

We find a tree $t \in T(\mathcal{F})$ and an accepting run $\rho \in T(Q)$, thereby showing that $L(\mathcal{A})$ is not empty. We define for this, for all $p \neq \varepsilon$, $t(p) := f \in \mathcal{F}_n$ for some n and $?(p) := q$ if $t'(p) = (q, f)$. By definition of δ', for any such position p, there exists a transition $(q, f, q_1, \dots, q_n) \in \delta$ such that $\rho(pi) = q_i$ for all $1 \leq i \leq n$. Regarding the root, we also know that there exists $q \in Q_f$ and $f \in \mathcal{F}_n$ for some n such that there is a transition $(q, f, q_1, \dots, q_n) \in \delta$ with $\rho(i) = q_i$ for all i, thus definining $t(\varepsilon) := f$ and $\rho(\varepsilon) := q$. Hence, there is a tree in $L(\mathcal{A})$ with an accepting run.

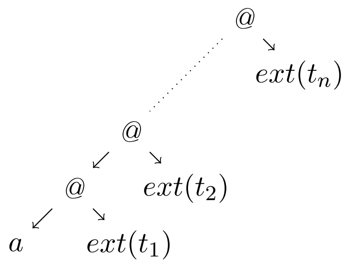
We deduce that emptiness for AWA on singleton alphabet is P-hard.

Exercise : Extensions

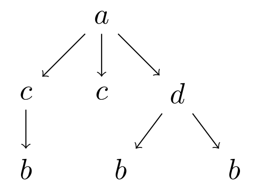
Definition 4 (extension encoding)
Let t be an unranked tree on Σ. Let $\mathcal{F}^{\Sigma}_{ext} = \{@(2)\} \cup \{a(0) \mid a \in \Sigma\}$. We define the ranked tree $ext(t)$ by induction on the size of t by :

- *for $a \in \Sigma$, $ext(a) = a$*
- *if $t = a(t_1, ..., t_n)$ with $n \geq 1$, $ext(t) = @(ext(a(t_1, ..., t_{n-1})), ext(t_n))$*

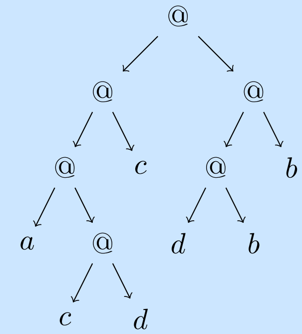
that is $ext(a(t_1, ..., t_n))$ is equal to :



1. Give the extension encoding of :



Solution:



2. Let L be a language of unranked trees. Prove that L is recognizable by a NFHA iff $ext(L)$ is recognizable by a NFTA.

Solution:

$\Rightarrow$) Let $\mathcal{A} = \langle Q, \Sigma, \Delta, F \rangle$ be a NFHA recognizing L such that there is exactly one rule of the form $a(L_{a,q}) \longrightarrow q$ for all (a, q) and let $B_{a,q} = \langle P_{a,q}, Q, p^0_{a,q}, \delta_{a,q}, F_{a,q} \rangle$ a deterministic automaton recognizing $L_{a,q}$. We construct the expected NFTA this way :

$$\mathcal{A}' = \langle Q', \mathcal{F}^{\Sigma}_{ext}, \Delta', F' \rangle$$

where :

- $Q' = \bigcup_{(a,q)} P_{a,q}$
- $F' = \bigcup_{(a,q) \mid q \in F} F_{a,q}$
- $\Delta' =$
  - ⋆ $a \longrightarrow p^0_{a,q}$ for all (a, q)
  - ⋆ $@(p, p') \longrightarrow p''$ if $p, p'' \in P_{b,q}$, $p' \in F_{a,q'}$ with $\delta_{b,q}(p, q') = p''$ for some b, q, a, q'

$\Leftarrow$) Let $\mathcal{A} = \langle Q, \mathcal{F}^{\Sigma}_{ext}, F, \Delta \rangle$ be a NFTA recognizing $ext(L)$. We construct the expected NFHA this way :

$$\mathcal{A}' = \langle Q, \Sigma, F, \Delta' \rangle$$

where for all (a, q), $a(R_{a,q}) \longrightarrow q \in \Delta'$ where $R_{a,q}$ is the language recognized by the automaton :

$$B_{a,q} = \langle Q, Q, I_{a,q}, F_{a,q}, \Delta_{a,q} \rangle$$

with :

- $I_{a,q} = \{p \in Q \mid a \longrightarrow p \in \Delta\}$
- $F_{a,q} =$
  - ⋆ $\{q\}$ if $q \in F$ or if there exists q', q'' such that $@(q', q) \longrightarrow q'' \in \Delta$
  - ⋆ $\varnothing$ else
- $\Delta_{a,q} = \{(q_1, q_2, q_3) \mid @(q_1, q_2) \longrightarrow q_3 \in \Delta\}$