Tree Automata and their Applications

TD n°3 : Logic and Hedges

2021-2022

Exercise 1 : MSO on finite trees

We consider trees with maximum arity 2. Give MSO formulae which express the following :

1. X is closed under predecessors
2. $x \subseteq y$ (with $\subseteq$ the prefix relation on positions)
3. 'a' occurs twice on the same path
4. 'a' occurs twice not on the same path
5. There exists a sub tree with only a's
6. The frontier word contains the chain 'ab'

Solution:

1. $closed(X) := \forall y \forall z (y \in X \wedge (z \downarrow_1 y) \vee z \downarrow_2 y)) \Rightarrow z \in X)$
2. $x \subseteq y := \forall X(y \in X \wedge closed(X) \Rightarrow X(x))$
3. $\exists x \exists y(\neg(x = y) \wedge x \subseteq y \wedge P_a(x) \wedge P_a(y))$
4. $\exists x \exists y(\neg(y \subseteq x) \wedge \neg(x \subseteq y) \wedge Pa(x) \wedge P_a(y))$
5. $\exists x \forall y(x \subseteq y \Rightarrow P_a(y))$
6. We first implement a way to say that a leaf is next to another one :

$$x \prec y := \exists x_0 \exists y_0 \exists z(z \downarrow_1 x_0) \wedge (z \downarrow_2 y_0) \wedge x_0 \subseteq x \wedge y_0 \subseteq y)$$

   And with this :

$$\exists x \exists y(Fr(x) \wedge Fr(y) \wedge P_a(x) \wedge P_b(y) \wedge x \prec y \wedge \neg \exists z(Fr(z) \wedge x \prec z \wedge z \prec y))$$

Exercise 2 : The power of Wsks

Produce formulae of WSkS for the following predicates :

- the set X has exactly two elements.
- the set X contains at least one string beginning with a 1.
- $x \leq_{lex} y$ where $\leq_{lex}$ is the lexicographic order on $\{1, ...k\}^*$.
- given a formula of WSkS ϕ with one free first-order variable, produce a formula of WSkS expressing that there is an infinity of words on $\{1, ..., k\}^*$ satisfying ϕ.

Solution:

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$$|X| \leq 2 \doteq \forall Y. Y \subseteq X \Rightarrow (Y = \varnothing \vee Sing(Y) \vee Y = X)$$
$$|X| \geq 2 \doteq \exists x, y. x \neq y \wedge x \in X \wedge y \in X$$
$$|X| = 2 \doteq |X| \leq 2 \wedge |X| \geq 2$$

- $$X \cap 1.\Sigma^* \neq \varnothing \doteq \exists x.\, x \in X \wedge 1 \leq x$$
- $$x \leq_{lex} y \doteq x \leq y \vee (\exists z. \bigvee_{i<j\leq k} (z.i \leq x \wedge z.j \leq y))$$
- $$X \models \phi \doteq \forall x, x \in X \Rightarrow \phi(x)$$
  $$\phi \text{ satisfied by an infinity of words } \doteq \forall X,\, X \models \phi \Rightarrow \exists Y,\, X \subsetneq Y \wedge Y \models \phi$$

Exercise 3 : The limit of Wsks

Prove that the predicate $x = 1y$ is not definable in WSkS.

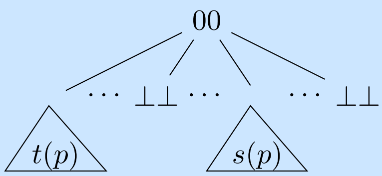
Solution:

We use the equivalence with recognizable tree languages. So we have to prove that $L = \{tra(x, y) \mid x = 1.y\}$ is not recognizable. Using the translation, we see that

$$L \cap \{t_i\sigma \mid t_i = 00(i\bot(x_1, ..., x_k), y_2, ..., y_k), i \in \{0, 1\}, \sigma \textit{ closed substitution}\}$$

$$= \{tra(x, y) \mid x = 1.y \wedge y \in \{2, ..., k\}.\{1, ..., k\}^*\} = L'$$

So it is enough to prove that L' is not recognizable. Now elements of L' are of the form :



with $p \in \{2, ..., k\}.\{1, ..., k\}^*$, t and s injective and the height of t and s strictly increasing with p. You can reason by contradiction using the pumping lemma : for p large enough, using the pumping lemma, you can iterate a piece of $t(p)$ without touching $s(p)$ (or vice versa) while staying in L' which is absurd by injectivity.

Exercise 4 : Alternating Word Automata

Definition 1 *If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi$.*

Definition 2 *A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subset Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word w is a tree t labelled by Q such that :*
— *if $w = \varepsilon$, then $t = q_0$ with $q_0 \in Q_0$.*
— *if $w = a.w'$, then $t = q_0(t_1, \ldots, t_n)$ $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, q_i, Q_f, \delta)$ and $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.*

Definition 3 *We say that a run is accepting if every leaf of the form q satisfies that $q \in Q_f$.*

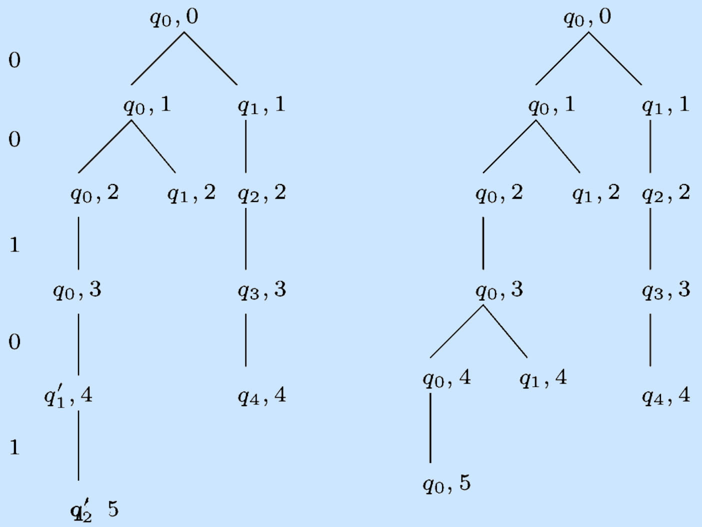
1. Let $\Sigma = \{0, 1\}$ and $\mathcal{A} = (Q, \Sigma, q_0, Q_f, \delta)$ the AWA such that $Q = \{q_0, q_1, q_2, q_3, q_4, q_1', q_2'\}$, $Q_f = \{q_0, q_1, q_2, q_3, q_4\}$ and :

$$\delta = \{ \quad \begin{array}{ll} q_0 0 \longrightarrow (q_0 \wedge q_1) \vee q_1' & q_0 1 \longrightarrow q_0 \\ q_1 0 \longrightarrow q_2 & q_1 1 \longrightarrow \top \\ q_2 0 \longrightarrow q_3 & q_2 1 \longrightarrow q_3 \\ q_3 0 \longrightarrow q_4 & q_3 1 \longrightarrow q_4 \\ q_4 0 \longrightarrow \top & q_4 1 \longrightarrow \top \\ q_1' 0 \longrightarrow q_1' & q_1' 1 \longrightarrow q_2' \\ q_2' 0 \longrightarrow q_2' & q_2' 1 \longrightarrow q_1' \} \end{array}$$

Give an example of an accepting computation of $\mathcal{A}$ on $w = 00101$ and an example of a non accepting computation of $\mathcal{A}$ on w.

2. Prove that for all AWA, we can compute in exponential time a non-deterministic automaton which accepts the same language.

Solution:



1. The left is non accepting but the right is.

2. Given $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ an AWA, we produce $\mathcal{A}' = (2^Q, \Sigma, 2^{Q_0}, 2^{Q_f}, \delta')$ with :

$$\delta'(S, a) = \{S' | S' \models \bigwedge_{s \in S} \delta(s, a)\}$$