Tree Automata and their Applications

TD n°3 : Logic and Hedges

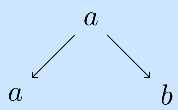
2021-2022

Exercise 1 : To the infinity...

Let $\Sigma = \{a, b\}$. Define a DFHA $\mathcal{A}$ such that $L(\mathcal{A})$ is the set of all trees such that "for every leaf labeled with a, there is an ancestor from which there is a path whose nodes are labeled with b". Here "ancestor" means strict ancestor and "from which there is a path" means that there is a path from a son of this ancestor to a leaf.

Solution:

Note : Many of you made the mistake of not considering trees of the following kind :



Any left of the tree that is labelled with a should have an ancestor (which can be labelled a or b) such that it has a branch that is completely labelled by bs. Hence, this is the following hedge automata. (Also, remember to have transitions from the leaves as well, a lot of you forgot that.)

$Q = \{q_a, q_b, q_\top\}$, $F = \{q_b, q_\top\}$ and $\Delta =$

- ⋆ $a(\epsilon) \longrightarrow q_a$
- ⋆ $a(q_\top^+) \longrightarrow q_\top$
- ⋆ $a((q_\top + q_a)^* q_a (q_\top + q_a)^*) \longrightarrow q_a$
- ⋆ $b(q_\top^+) \longrightarrow q_\top$
- ⋆ $b(\epsilon) \longrightarrow q_b$
- ⋆ $a(Q^* q_b Q^*) \longrightarrow q_\top$
- ⋆ $b(Q^* q_b Q^*) \longrightarrow q_b$
- ⋆ $b((q_\top + q_a)^* q_a (q_\top + q_a)^*) \longrightarrow q_a$

⋆⋆⋆

Exercise 2 : Membership

1. Recall the complexity of the uniform membership problem for DFTAs, NFTAs and NFHAs.
2. Prove that **(AlternatingUMembership)** :
   Instance : an AWA $\mathcal{A}$ and a word w
   Question : $w \in L(\mathcal{A})$?
   is in PTIME.

3. Prove that **(HarderUMembership)** :
   Instance : an NFHA $\mathcal{A}$ where the horizontal languages are given by AWA (and not finite automata) and a word w
   Question : $w \in L(\mathcal{A})$?
   is in NP.

4. Let Φ be a propositional formula in CNF with n variables $x_1, ..., x_n$. Construct, in polynomial time, an AWA $\mathcal{A}_\Phi$ whose language is $\{w \in \{0,1\}^n \mid w \models \Phi$ i.e. $\Phi_{[x_i \leftarrow w_i]} = \top\}$.
5. Deduce that membership for NFHA where horizontal languages are given by AWA is NP-complete.

Solution:

1. In the case of DFTA, from a term t and the automaton A, we can compute a run in $O(|t| + |A|)$. In the nondeterministic case, the idea is similar to the word case : the algorithm determinizes along the computation, i.e. for each node of the term, we compute the set of reached states. The complexity of this algorithm will be in $O(|t| \times |A|)$.
   For NFHAs, the idea is similar to the NFTA, and we obtain a PTIME algorithm.
2. Let n be the size of w. Define S_i with $0 \leq i \leq n$ by decreasing induction :
   — $S_n = Q_f$
   — $S_i = \{q \in Q \mid \exists q, w_i \rightarrow \phi \in \mathcal{A}, S_{i+1} \models \phi\}$
   Then $w \in L(A)$ iff $I \cap S_0 \neq \varnothing$. All this can be done in PTIME.
3. - in NP : first, guess a run i.e. a coloring ρ by states of your tree. Second, check this is an accepting run i.e. for all position p of t, check that there exists a transition of the form $t(p)(L) \longrightarrow \rho(p)$, that $\rho(p.1)...\rho(p.k) \in L$ (which can be done in P by exercise 1) and $\rho(\epsilon)$ is a final state.
   - NP-hard : we reduce SAT. Let Φ in CNF. We construct $\tilde{\mathcal{A}}_\Phi$ this way : $Q = \{0, 1, q_f\}$, $F = \{q_f\}$, $\Sigma = \{@, \#\}$ and $\Delta = \{\#(\epsilon) \longrightarrow 1, \#(\epsilon) \longrightarrow 0, a(\mathcal{A}_\Phi) \longrightarrow q_f\}$ where $\mathcal{A}_\Phi$ is from exercise 2. Then Φ is satisfiable iff $a(\#, ..., \#) \in L(\tilde{\mathcal{A}}_\Phi)$.
4. Let $\Phi = \bigwedge_{j=1}^{m} C_j$ with C_j clauses. The expected AWA is the following :
   - $Q = \{q_0\} \cup \{q_j^{\alpha,k} \mid 1 \leq j \leq m, 1 \leq k \leq n, \alpha \in \{0,1\}\}$
   - $I = \{q_0\}$
   - $F = \{q_j^{1,n} \mid 1 \leq j \leq m\}$
   - $\Delta =$
     - ⋆ $q_0, 1 \longrightarrow \bigwedge_{j \mid x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid x_1 \notin C_j} q_j^{0,1}$
     - ⋆ $q_0, 0 \longrightarrow \bigwedge_{j \mid \neg x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid \neg x_1 \notin C_j} q_j^{0,1}$
     - ⋆ $q_j^{1,k}, \_ \longrightarrow q_1^{1,k+1}$ for all $k < n$ and all j
     - ⋆ $q_1^{0,k}, 1 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $x_k \in C_j$
     - ⋆ $q_1^{0,k}, 1 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $x_k \notin C_j$
     - ⋆ $q_1^{0,k}, 0 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $\neg x_k \in C_j$
     - ⋆ $q_1^{0,k}, 0 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $\neg x_k \notin C_j$
5. NP-hard : we reduce SAT. Let Φ in CNF. We construct $\tilde{\mathcal{A}}_\Phi$ this way : $Q = \{0, 1, q_f\}$, $F = \{q_f\}$, $\Sigma = \{a, \#\}$ and $\Delta = \{\#(\epsilon) \longrightarrow 1, \#(\epsilon) \longrightarrow 0, a(\mathcal{A}_\Phi) \longrightarrow q_f\}$ where $\mathcal{A}_\Phi$ is from the previous question. Then Φ is satisfiable iff $a(\#, ..., \#) \in L(\tilde{\mathcal{A}}_\Phi)$.